Ahmad pairs and the local structure of the enumeration degrees

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The Enumeration Degrees

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We say $X \leq_e Y$ if every enumeration of Y computes an enumeration of X.

A more useful characterization for constructions is the following: $X \leq_e Y$ if there is a c.e. operator Γ such that $x \in X \iff \exists \langle x, D \rangle \in \Gamma$ with $D \subseteq Y$.

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1. This is a positive analog of Turing reducibility. One can think of $X \leq_T Y$ as X is $\Delta^0_1(Y)$. Here we can think of $X \leq_e Y$ as X is $\Sigma^+_1(Y)$, so we can only ask positive questions about Y.

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2. $\leq_e$ is a preorder on $\mathcal{P}(\omega)$ and the corresponding degree structure induced on $\mathcal{P}(\omega)$ is called the enumeration degrees denoted by $(\mathcal{D}_e, \leq_e)$.
3. The enumeration jump of a set X is defined by $X' = K_X \oplus \overline{K}_X$ where $K_X := \bigoplus_{n\in\omega} \Gamma_n(X)$. The skip of a set is defined as $X^\diamond = \overline{K_X}$. While these two operations do not agree in general, they will agree for the degrees we care about in this talk.

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5. The Turing degrees $(\mathcal{D}_T, \leq_T)$ embed into the enumeration degrees as a partial order under the map $\iota : A \to A \oplus \overline{A}$ and this embedding respects join and jump. The image of this mapping are the total degrees: degrees containing a set X for which $\overline{X} \leq_e X$.

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6. $\mathcal{D}_T$ and $\mathcal{D}_e$ are both upper semilattices with a least element. The least element in D_T is the family of computable sets while in $\mathcal{D}_e$ the least element 0_e is made up of the c.e. sets.

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7. Monotonicity is a salient property of enumeration operators : If $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$ for any enumeration operator Γ. Note that no such simple relation holds in the Turing world.

Local Structure

Let $\mathcal{D}_T(\leq 0'_T)$ be the Turing degrees below $0'_T$ while $\mathcal{D}_e(\leq 0'_e)$ denotes the enumeration degrees below $0'_e$. Both of these sets form a countable ideal in their respective degree structures.

The standard representative of $0'_e$ is the Π^0_1 complete set $\overline{K} = \{e : e \notin \Gamma_e(\emptyset)\}$.

1. $X \leq_T 0'_T$ iff X is Δ^0_2.
2. On the other hand, $X \leq_e 0'_e$ iff X is Σ^0_2 iff X is c.e. relative to $0'$.
3. We have the relation : $\iota(\mathcal{D}_T(\leq 0'_T)) \subset D_e(\leq 0'_e)$.

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3. We have the relation : $\iota(\mathcal{D}_T(\leq 0'_T)) \subset D_e(\leq 0'_e)$.
4. These two structures are not elementarily equivalent as partial orders, and they have several different structural properties.
5. One such elementary difference is the density of $\mathcal{D}_e(\leq 0'_e)$. Cooper [Coo84] was able to show that given $X <_e Y \leq 0'_e$, there is a Z such that $X <_e Z <_e Y$.
6. In particular, there are no minimal degrees or minimal covers. Full density does not extend to the global structure, but Guttridge showed that it is *downwards* dense.

First order theory of the local structure

The extension of embeddings problem asks: For finite partial orders $\mathcal{P} \subset \mathcal{Q}$, when can every embedding of $\mathcal{P}$ be extended to an embedding of $\mathcal{Q}$. This is a proper subproblem of the $\forall\exists$ theory of $\mathcal{D}_e(\leq 0'_e)$ in the language of $\{\leq_e\}$.

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1. Slaman and Soare [SS01] solved the extension of embeddings problem in the c.e. Turing degrees: They provide two obstructions barring which an extension is always possible.
2. Lempp, Slaman and Sorbi [LSS05] solve the extension of embeddings problem for the Σ_2^0 enumeration degrees, where the only *added obstruction to extension is the phenomenon of Ahmad pairs.*
3. Kent [Ken06] showed that the $\forall\exists\forall$ theory of $\mathcal{D}_e(\leq 0_e')$ is undecidable.

Recently there has been a renewed interest in solving the $\forall\exists$ theory of the Σ_2^0 enumeration degrees. With several papers extending the frontier in different directions, there has been a renewed interest in Ahmad pairs.

Ahmad pairs

Definition

Let A, B be Σ^0_2 sets. Then (A, B) is an Ahmad pair if $A \not\leq_e B$ and $\forall Z <_e A (Z <_e B)$.

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Theorem (Ahmad [AL98])

There are Δ^0_2 sets A and B such that (A, B) is an Ahmad pair. Moreover if (A, B) is an Ahmad pair, then (B, A) is not an Ahmad pair.

1. Note that A must be join irreducible, and $\{Z : Z <_e A\}$ is an ideal with (A, B) as an exact pair.

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1. Note that A must be join irreducible, and $\{Z : Z <_e A\}$ is an ideal with (A, B) as an exact pair.
2. This phenomenon cannot occur in the c.e. Turing degrees due to Sacks splitting.
3. This can happen in the $\mathcal{D}_T(\leq_T 0')$: Any strong minimal cover can be made the left half of an Ahmad pair. However it is more interesting in $\mathcal{D}_e(\leq_e 0'_e)$ due to its density.

More on Ahmad pairs

The following work is more recent.

Theorem (Goh, Lempp, Ng, M. Soskova[Goh+22])

If (A, B) is an Ahmad pair, then (B, C) is not an Ahmad pair for any C.

Theorem (Kalimullin, Lempp, Ng, Yamaleev[Kal+24])

If (A, B) is an Ahmad pair, then $A \oplus B <_e 0'_e$.

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I was able to show that Ahmad pairs have interesting interactions with jump classes.

Theorem

The following are equivalent for a set A:

1. *The family $\{Y : Y <_e A\}$ has a uniform enumeration relative to $0'_e$.*
2. $X = \{n : \Gamma_n(A) <_e A\}$ *is* Σ^0_4.
3. A *is* low_3.

We can now characterize the left half.

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For a computable function f, we call the family $\{\Gamma_{f(n)}(C)\}_n$ an *Ahmad sequence for A relative to C* if $\{\Gamma_{f(n)}(C)\}_n = \{Z : Z <_e A\}$.

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The following are equivalent:

1. *A has an Ahmad sequence $\{\Gamma_{f(n)}(C)\}_n$ relative to C.*
2. *A is low_3, C is high_2 and either $A \leq_e C$ or (A, C) is an Ahmad pair.*

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2. *A is low_3, C is high_2 and either $A \leq_e C$ or (A, C) is an Ahmad pair.*

Lemma

Suppose (A, B) is an Ahmad pair. Then A has an Ahmad sequence relative to B.

Corollary

If (A, B) is an Ahmad pair, then B is high_2.

Extending Join Irreducibility

Definition

A is n-join irreducible if for every $A_0, \ldots, A_n <_e A$ there is an $i, j \leq n$ with $A_i \oplus A_j <_e A$.

This can be shown to be equivalent to $\{Z : Z <_e A\}$ having a covering by n-ideals i.e. $\{Z : Z <_e A\} = \bigcup_1^n \mathcal{F}_i$ where each $\mathcal{F}_i$ is an ideal.
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Theorem

For every $n \geq 0$, there is a low set A which is n-join reducible and $n+1$-join irreducible.

Definition

The tuple $(A, B_0, ..., B_{n-1})$ is an Ahmad n-pair if $A \not\leq_e B_i$ for every $i < n$ and any $Z <_e A$ is below at least one of the B_i's.

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Theorem

For every $n > 1$, there is an Ahmad n-pair whose left half is not part of an Ahmad $(n-1)$-pair.

Other Interesting Properties

The following result shows that join irreducibility is downwards dense in $\mathcal{D}_e(0'_e)$.

Theorem (Kent, Sorbi [KS07])

For every $X >_e 0$, there is a $0 <_e Y \leq_e X$ such that Y is join irreducible.

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This fascinating result show that below every Σ_2 set X, is an initial segment all of whose elements have the same jump as X.

Theorem (Ganchev, Sorbi [GS16])

For any set C, there is an initial segment $\mathcal{I} = \{X : 0 <_e X \leq_e D\}$ for some non-c.e. $D \leq_e C$, such that for every $X \in \mathcal{I}$, $X' = C'$.

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Kalimulin defined $\mathcal{K}$-pairs, which is probably the single most useful tool in the study of the enumeration degrees. $\{X, Y\}$ for a $\mathcal{K}$-pair if $X \times Y$ and $\overline{X \times Y}$ can be separated by a c.e. set.

Theorem

$\{X, Y\}$ form a $\mathcal{K}$-pair if and only if $\forall Z (X \vee Z) \wedge (Y \vee Z) \equiv_e Z)$.

Jump Classes

Definition

Let $n \in \mathbb{N}$, $n \geq 1$ and let X be Σ^0_2.

1. X is low$_n$ if $X^{(n)} \leq_e 0^{(n)}$.
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1. Using coding via first order arithmetic, all these jump classes for the Δ^0_2 Turing degrees were shown to be definable, with the exception of low (still open).
2. In the enumeration setting, the jump operation was shown to be definable by Kalimullin [Kal03] using $\mathcal{K}$-pairs. This was extended to the local setting by Ganchev and Soskova [GS15] to show among other things, that low was first order definable in the local structure (low $\iff$ everything below bounds a $\mathcal{K}$ pair).
3. The same authors [GS18] later showed that all jump classes are definable by going through definability of the corresponding classes in the Turing degrees, and hence these definitions have high quantifier complexity

Defining low_3 and high_2

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The definition below says that a degree is low_3 precisely when everything non-trivial below it bounds a left half.

Theorem

A is low_3 iff $\forall Z \leq_e A(\exists Y <_e Z \implies \exists Y <_e Z, \exists B \not\geq_e Y(\forall X <_e Y(X \leq_e B)))$.

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The definition below says that X is high_2 precisely when everything above it which bounds a left half also bounds a right half.

Theorem

X is high_2 if and only if $\forall Y \geq_e X \forall A \leq_e Y(\exists B \not\geq_e A \forall Z <_e A(Z \leq_e B)) \implies (\exists C \leq_e Y, C \not\geq_e A \forall Z <_e A(Z \leq_e C))$.

For more details on the proofs of these theorems: [Rav25]

Some open questions

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The following has been very well studied in the Turing world. However I have not found any results implying that it is possible in the enumeration world. Cupping would imply join irreducibility is not upwards dense.

Question

X has the cupping property if $\forall Y > X \exists Z < Y (Z \oplus X \equiv Y)$. Are there any degrees with the cupping property in the enumeration degrees?

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Our definitions of low_3 and high_2 essentially go through the index set of incomplete indices. Can any other index set be used to analyze other jump classes.

Question

Do any other jump classes have simpler definitions without going through the Turing jump?

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