Defining low_3, high_2 in the Σ_2 e-degrees

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The Enumeration Degrees

Definition

We say $X \leq_e Y$ if every enumeration of Y computes an enumeration of X.

A more useful characterization for constructions is the following: $X \leq_e Y$ if there is a c.e. operator Γ such that $x \in X \iff \exists\langle x, D\rangle \in \Gamma$ with $D \subseteq Y$.

1. This is a positive analog of Turing reducibility. One can think of $X \leq_T Y$ as X is $\Delta^0_1(Y)$. Here we can think of $X \leq_e Y$ as X is $\Sigma^+_1(Y)$, so we can only ask positive questions about Y.
2. $\leq_e$ is a preorder on $\mathcal{P}(\omega)$ and the corresponding degree structure induced on $\mathcal{P}(\omega)$ is called the enumeration degrees denoted by $(\mathcal{D}_e, \leq_e)$.
3. The enumeration jump of a set X is defined by $X' = K_X \oplus \overline{K}_X$ where $K_X := \bigoplus_{n\in\omega} \Gamma_n(X)$. The skip of a set is defined as $X^\diamond = \overline{K_X}$. While these two operations do not agree in general, they will agree for the degrees we care about in this talk.

The Enumeration Degrees

4. Observe : $X \leq_T Y$ iff $X \oplus \overline{X} \leq_e Y \oplus \overline{Y}$. So we can think of Turing reducibility in terms of enumeration reducibility.

5. The Turing degrees $(\mathcal{D}_T, \leq_T)$ embed into the enumeration degrees as a partial order under the map $\iota : A \to A \oplus \overline{A}$ and this embedding respects join and jump. The image of this mapping are the total degrees: degrees containing a set X for which $\overline{X} \leq_e X$.

6. $\mathcal{D}_T$ and $\mathcal{D}_e$ are both upper semilattices with a least element. The least element in D_T is the family of computable sets while in $\mathcal{D}_e$ the least element 0_e is made up of the c.e. sets.

7. Monotonicity is a salient property of enumeration operators : If $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$ for any enumeration operator Γ. Note that no such simple relation holds in the Turing world.

Local Structure

Let $\mathcal{D}_T(\leq 0'_T)$ be the Turing degrees below $0'_T$ while $\mathcal{D}_e(\leq 0'_e)$ denote the enumeration degrees below $0'_e$. Both of these sets form a countable ideal in their respective degree structures. The standard representative of $0'_e$ is the Π^0_1 complete set $\overline{K} = \{e : e \notin \Gamma_e(\emptyset)\}$.

1. $X \leq_T 0'_T$ iff X is Δ^0_2 while $X \leq_e 0'_e$ iff X is Σ^0_2 iff X is c.e. relative to $0'$.
2. We have the relation : $\iota(\mathcal{D}_T(\leq 0'_T)) \subset D_e(\leq 0'_e)$.
3. These two structures are not elementarily equivalent as partial orders: For example, Sacks constructed a Δ^0_2 minimal Turing degree while Gutridge showed that the enumeration degrees are downward dense (globally).
4. In fact Cooper [Coo84] was able to show that given $X <_e Y \leq 0'_e$, there is a Z such that $X <_e Z <_e Y$. So $\mathcal{D}_e(\leq 0'_e)$ is actually dense.
5. This is far from true in the global structure: Calhoun and Slaman [CS96] construct an empty interval in the Π^0_2 enumeration degrees.

Ahmad pairs

Definition

Let A, B be Σ^0_2 sets. Then (A, B) is an Ahmad pair if $A \not\leq_e B$ and $\forall Z <_e A (Z <_e B)$.

This is almost like a strong minimal cover, except $A \not\geq_e B$.

Theorem (Ahmad [AL98])

There is an Ahmad pair (A, B) with A, B - Δ^0_2. Moreover if (A, B) is an Ahmad pair, then (B, A) is not an Ahmad pair.

This phenomenon cannot occur in the c.e. Turing degrees due to Sacks splitting, and hence the c.e. Turing degrees are not elementarily equivalent to $\mathcal{D}_e(\leq 0'_e)$. The following work is more recent.

Theorem (Goh, Lempp, Ng, M. Soskova[Goh+22])

If (A, B) is an Ahmad pair, then (B, C) is not an Ahmad pair for any C.

Theorem (Kalimullin, Lempp, Ng, Yamaleev[Kal+24])

If (A, B) is an Ahmad pair, then $A \oplus B <_e 0'_e$.

Importance of Ahmad pairs

1. Slaman and Soare [SS01] solved the extension of embeddings problem in the c.e. Turing degrees: For finite partial orders $\mathcal{P} \subset \mathcal{Q}$, when can every embedding of $\mathcal{P}$ be extended to an embedding of $\mathcal{Q}$. They provide two obstructions barring which an extension is always possible.
2. Lempp, Slaman and Sorbi [LSS05] solve the extension of embeddings problem for the Σ_2^0 enumeration degrees, where the only *added obstruction to extension is the phenomenon of Ahmad pairs.*
3. Kent [Ken06] showed that the $\forall\exists\forall$ theory of $\mathcal{D}_e(\leq 0_e')$ is undecidable.

Recently several researchers have been focusing on solving the $\forall\exists$ theory of the Σ_2^0 enumeration degrees. This has resulted in renewed interest in Ahmad pairs. Previous work separating the right and left half [Goh+22] was quite involved, using a $0^{'''}$ priority construction. Moreover, it did not explain why the two halves are separate. We present a simpler proof of a more general result, which gives a natural separation between the two halves in terms of jump classes.

New Work on Ahmad pairs

Note that if (A, B) is an Ahmad pair, then the set $\{Z : Z <_e A\}$ is an ideal. In particular A is join irreducible. Observe that (A, B) form an exact pair for this ideal.

Theorem

The following are equivalent for a set A:

1. *The family $\{Y : Y <_e A\}$ has a uniform enumeration*
2. $X = \{n : \Gamma_n(A) <_e A\}$ *is* Σ^0_4.
3. A *is* low_3.

We can generalize this to Ahmad n-pairs.

Definition

1. The tuple $(A, B_0, ..., B_{n-1})$ is an Ahmad n-pair if $A \not\leq_e B_i$ for every $i < n$ and any $Z <_e A$ is below at least one of the B_i's.
2. A is n join irreducible if for every $A_0, \ldots, A_n <_e A$ there is an $i, j \leq n$ with $A_i \oplus A_j <_e A$.

On the left and right halves

Theorem

A is the left half of an Ahmad n-pair $\iff$ A is low_3 and n join irreducible.

Having characterized the left halves, we can now turn our attention to the right half.

Lemma

Suppose (A, B) is an Ahmad pair. Then $A' \leq_e B'$.

Definition

For a computable function f, we call the family $\{\Gamma_{f(n)}(C)\}_n$ an *Ahmad sequence for A relative to C* if $\{\Gamma_{f(n)}(C)\}_n = \{Z : Z <_e A\}$.

Theorem

The following are equivalent:

1. *A has an Ahmad sequence $\{\Gamma_{f(n)}(C)\}_n$ relative to C.*
2. *A is low_3, C is high_2 and either $A \leq_e C$ or (A, C) is an Ahmad pair.*

On the right half

Lemma

1. *Suppose (A, B) is an Ahmad pair. Then A has an Ahmad sequence relative to B.*
2. *Conversely if A has an Ahmad sequence relative to C, then $\exists B <_e C$ such that (A, B) is an Ahmad pair.*

Using this, we obtain the desired separation.

Corollary

If (A, B) is an Ahmad pair, then B is high_2.

Using a different analysis, we can in fact extend this to Ahmad n-pairs as well.

Corollary

if $(A, B_0, \ldots, B_n)$ is an Ahmad $(n+1)$-pair and $(A, B_0, \ldots, B_{n-1})$ is not an Ahmad n-pair, then B_n is high_2.

We now turn our attention to definability of the corresponding jump classes. We will end up using several other interesting theorems known about the Σ_2 e-degrees in showing our definition works.

Jump Classes

Definition

Let $n \in \mathbb{N}$, $n \geq 1$ and let X be Σ^0_2.

1. X is low$_n$ if $X^{(n)} \leq_e 0^{(n)}$.
2. X is high$_n$ if $X^{(n)} \geq_e 0^{(n+1)}$.

Using coding via first order arithmetic, all these jump classes for the Δ^0_2 Turing degrees were shown to be definable, with the exception of low (still open).
In the enumeration setting, the jump operation was shown to be definable by Kalimullin in [Kal03] using what were later named $\mathcal{K}$-pairs. This was extended to the local setting by Ganchev and Soskova [GS15] to show among other things, that low was first order definable in the local structure.
The same authors [GS18] later showed that all jump classes are definable by going through definability of the corresponding classes in the Turing degrees, and hence these definitions have high quantifier complexity

Defining low_3

Theorem (Kent, Sorbi [KS07])

For every $X >_e 0$, there is a $0 <_e Y \leq_e X$ such that Y is join irreducible.

Theorem (Ganchev, Sorbi [GS16])

For any set C, there is an initial segment $\mathcal{I} = \{X : 0 <_e X \leq_e D\}$ for some non-c.e. $D \leq_e C$, such that for every $X \in \mathcal{I}$, $X' = C'$.

Combining these two facts with our analysis, we get the following Π_3 definition of low_3 in the language of $\leq_e$.

Theorem

A is low_3 iff $\forall Z \leq_e A(\exists Y <_e Z \implies \exists Y <_e Z, \exists B \not\geq_e Y(\forall X <_e Y(X \leq_e B)))$.

Proof.

Suppose A is low_3. Then for every $Z \in (0_e, A)$ there is a Y which is join irreducible. Then Y is the left half of an Ahmad pair.
On the other hand if A is not low_3 then there is a $Z \leq_e A$ such that $\forall Y \in (0_e, Z)$ we have $Y' \equiv_e A'$ i.e. Y is not low_3. Then Y cannot be the left half of an Ahmad pair. □

Defining high_2

Theorem (Kalimullin [Kal03])

For every X, there is a low set A such that $(X \oplus A)' \equiv X'$.

Proof.

Let $X >_e 0_e$, and let (A, B) be a $\mathcal{K}$-pair below $0_e'$ with $A, B \not\leq_e X$. Then A is low and $(A \oplus X)' \leq_e A \oplus B \oplus X'$. Since A and B are Σ^0_2, $(A \oplus X)' \leq_e X'$. □

The definition below says that X is high_2 precisely when everything above it which bounds a left half also bounds a right half.

Theorem

X *is* high_2 *if and only if* $\forall Y \geq_e X \forall A \leq_e Y (\exists B \not\geq_e A \forall Z <_e A (Z \leq_e B)) \implies (\exists C \leq_e Y, C \not\geq_e A \forall Z <_e A (Z \leq_e C))$.

Proof.

If X is high_2 and $Y \geq_e X$, then Y is high_2. Therefore if Y bounds a left half, then it also bounds a right half corresponding to that left half.
If X is not high_2, there is a $Y \equiv_e X \oplus A$ which bounds a left half as it bounds a low join irreducible degree [KS07]. However Y cannot bound a right half. □

Unanswered questions

We know the following about right halves:

1. There is a right half B which is high_2 but not high.
2. There is no maximal or minimal right half.
3. There is a high degree which does not bound any right half.

We end with some questions.

Question

Is there a characterization of the right halves of an Ahmad pair? Can they be join irreducible?

Question

If $(A, B_0, \ldots, B_{n-1})$ is an Ahmad n-pair, then is $A \oplus B_i <_e 0'_e$? Is there a simpler explanation in this framework for the non cupping of Ahmad pairs?

For more details on the proofs of the theorems: [Rav25]

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