Enumerations and Computing Generics

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The Enumeration Degrees

Definition

We say $X \leq_e Y$ if every enumeration of Y computes an enumeration of X.

A more useful characterization for constructions is the following: $X \leq_e Y$ if there is a c.e. operator Γ such that $x \in X \iff \exists \langle x, D\rangle \in \Gamma$ with $D \subseteq Y$.

1. This is a positive analog of Turing reducibility. One can think of $X \leq_T Y$ as X is $\Delta^0_1(Y)$. Here we can think of $X \leq_e Y$ as X is $\Sigma^+_1(Y)$, so we can only ask positive questions about Y.
2. $\leq_e$ is a preorder on $\mathcal{P}(\omega)$ and the corresponding degree structure induced on $\mathcal{P}(\omega)$ is called the enumeration degrees denoted by $(\mathcal{D}_e, \leq_e)$.
3. The enumeration jump of a set X is defined by $X' = K_X \oplus \overline{K}_X$ where $K_X := \bigoplus_{n\in\omega} \Gamma_n(X)$. The skip of a set is defined as $X^\diamond = \overline{K_X}$. While these two operations do not agree in general, they agree on a large class of degrees called the cotoal degrees.

The Enumeration Degrees

4. Observe : $X \leq_T Y$ iff $X \oplus \overline{X} \leq_e Y \oplus \overline{Y}$. So we can think of Turing reducibility in terms of enumeration reducibility.

5. The Turing degrees $(\mathcal{D}_T, \leq_T)$ embed into the enumeration degrees as a partial order under the map $\iota : A \to A \oplus \overline{A}$ and this embedding respects join and jump. The image of this mapping are the total degrees: degrees containing a set X for which $\overline{X} \leq_e X$.

6. $\mathcal{D}_T$ and $\mathcal{D}_e$ are both upper semilattices with a least element. The least element in D_T is the family of computable sets while in $\mathcal{D}_e$ the least element 0_e is made up of the c.e. sets.

7. Monotonicity is a salient property of enumeration operators : If $A \subseteq B$ then $\Gamma(A) \subseteq \Gamma(B)$ for any enumeration operator Γ. Note that no such simple relation holds in the Turing world.

Minimal Degrees

We say that X is minimal if $Y < X \implies Y \equiv 0$. Spector showed that minimal Turing degrees exist.

Theorem (Gutteridge)

There is an enumeration operator Θ such that for every $A \subseteq \omega$ either $0_e <_e \Theta(A) <_e A$ or A is Δ^0_2.

The same can be shown using a different argument for the Δ^0_2 degrees. Therefore the enumeration degrees are (almost uniformly) downwards dense and cannot have minimal degrees. This is one of the many elementary differences between the two structures.

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The idea is that X is minimal with respect to the total degrees. Quasiminimals exist in great abundance both in the sense of measure and category. Every 1-generic is quasiminimal and every weak 2-random is quasiminimal.

Importance of total degrees

Every enumeration degree is completely characterized by the total degrees above it. This is captured in Selman's theorem below.

Theorem (Selman's Theorem)

The following are equivalent.

1. $A \leq_e B$.
2. *Every enumeration of* B *uniformly computes an enumeration of* A.
3. $\{X : B \text{ is c.e. in } X\} \subseteq \{X : A \text{ is c.e. in } X\}$.

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It is natural to analyze the interaction between quasiminimal degrees and the total degrees above them.

Theorem (Slaman, Sorbi [SS98])

There is a quasiminimal degree with continuum many minimal Turing degrees above it.

Such a degree must in a sense be low down.

Cones above quasiminimals

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Theorem (R.)

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Notions of Computational Strength

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Let $f : \omega \to \omega$ be a function and $\mathcal{F} \subseteq \omega^\omega$ a family of functions. We say f escapes $\mathcal{F}$ if $\forall g \in \mathcal{F} \exists^\infty n \;\; f(n) > g(n)$.

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Definition

Given a set of strings W, we say $X \in 2^\omega$ meets W if $\exists \sigma \prec X$ such that $\sigma \in W$.
We say X avoids W if $\exists \sigma \prec X$ such that$\forall \tau \succ \sigma (\tau \notin W)$.
We say X is weak n-generic if it meets every dense Σ^0_n set of strings and that X is n-generic if it meets or avoids every Σ^0_n set of strings.

Note that n-generic implies weak n-generic which in turn implies $(n-1)$-generic for every $n > 1$.

Notions of Computation Strength

The table below summarizes the relationship between levels of genericity and whether they are characterized by a certain notion of computational strength or if they are implied by certain notions of computational strength.

	Escaping	Infinite subset	Enumeration
Weakly 1-generic	Yes	Yes	Yes
1-generic	Sufficient	Sufficient	Sufficient
Weakly 2-generic	Yes	Yes	Yes
2-generic	Sufficient	Sufficient	Sufficient
Weakly 3-generic	No	No	Sufficient*

1. It is easy to see that if X is weakly n-generic, then p_X escapes all $0^{(n-1)}$ computable functions.
2. Given f which escapes all computable functions, we can use it to compute a weak 1-generic by meeting all dense sets of strings. This explains row 1.

Characterizing genericity

3. Andrews, Gerdes and Miller [AGM14] show that if f escapes all Δ^0_2 functions, then f computes a weak 2-generic. This explains Row 3.
4. In the same paper, the authors show that no amount of escaping power ensures that one can compute a weak 3-generic.
5. Extending those ideas, I could argue that every weak 3-generic has an infinite subset which computes no weak 3-generic.
6. The authors in [AGM14] also show that if f escapes all Δ^0_3 functions, then f computes a 2-generic.

Enumerations

Lemma (R.)

For any n, enumeration power cannot characterize n-genericity.

Proof.

The authors in [Bad+16] show that every Δ^0_2 set enumerates a 1-generic. Relativizing this, we can see that every Δ^0_{n+1} set which is above $0^{(n-1)}$ can enumerate a n-generic. However not all such sets compute a n-generic: Take a minimal cover M of $0^{(n-1)}$. If $X_0 \oplus X_1 \leq_T M$ was n-generic, then X_0 would be X_1 n-generic. But since $X_1 \oplus 0^{(n-1)} \equiv_T M$ we would have X_0 is M-generic! $\square$

So now the question becomes if enumeration power is sufficient to ensure computing a sufficiently generic.

Definition

We say X is fat if $\forall n$ we have $2n \in X \vee 2n+1 \in X$.

Enumerations of fat sets are useful because we know we can always wait for either $2n$ or for $2n+1$ to appear.

Weak infinite subsets

Lemma

Every weak 3-generic has an infinite subset which computes no weak 3-generic.

Proof.

We build a $0''$ computable function tree $f : \omega^{<\omega} \to \omega^{<\omega}$ such that none of the paths through f compute a weak 3-generic. Once we define $f(\tau)$, we associate a c.e. set of possible extensions V_τ to it, starting with $V_{\tau,0} = \{f(\tau)m : m \in \omega\}$.
We simultaneously build a Σ^0_3 dense set of strings U_e and ensure that no prefix of $\varphi_e^{f(\tau)}$ is in U_e.
To make U_e dense, we add at stage $s = \langle k, i\rangle$, extensions of every string at level k into U_e. To do this, we ask if $\exists \rho \in 2^k \exists^\infty \gamma \in V_\tau$ such that each γ has an extension $\hat{\gamma} \succ \gamma$ such that $\varphi_e^{\hat{\gamma}} = \rho$. If yes, we set $V_{\tau,s+1}$ to be the set of those extensions. Otherwise ρ is undefined. We extend U_e by all strings at level 2^k except the ρ's corresponding to $e < s$.
Since it is dense to be a superset of some extension of $f(\tau)$ for any τ (due to the fact that there are large enough extensions and that each path is increasing), we can then have room to find a subset with the desired property. □

Fat generics

Theorem

For any $X \subseteq \omega$, there is a quasiminimal e-degree such that all the total degrees above it compute a X 1-generic

Proof.

We shall build a fat set Y with this property. We build Y in stages as $Y = \bigcup Y_s$.
Let $Y_0 = \emptyset$.
At stage $2s+1$ we check if there is a fat $\tau \succ Y_{2s}$ such that $2n, 2n+1 \in \Gamma_s^\tau$ for some n. If such a τ exists, we set $Y_{2s+1} = \tau$, ensuring that Γ_s^Y is not total.
Otherwise let $Y_{2s+1} = Y_{2s}$. Then if Γ_s^Y is total with Y fat, say $A \oplus \overline{A} = \Gamma_s^Y$, we can argue A is computable: $n \in A \iff \exists \tau \succ Y_{2s}$ with τ fat and $2n \in \Gamma_s^\tau$ and $n \notin A \iff \exists \tau \succ Y_{2s}$ with τ fat and $2n+1 \in \Gamma_s^\tau$. (We are using the fact that union of fat strings is fat)
At stage $2s+2$, we proceed as follows: let n be sufficiently large, and let $N > n$ be such that each string $\sigma \in 2^n$ has an extension $\tau_\sigma \in 2^N$ which meets W_s^X or avoids W_s^X. Then we define Y_{2s+2} to be 01 or 10 at the appropriate bits to encode the correct extension to take.

□

Minimal pairs

The above result shows that if we have any comeager upwards closed class of Turing degrees, then we can find a quasiminimal whose cone only contains total sets from this class.

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Lemma

Let A be 2-generic relative to B. Then A and B form a Σ_1 minimal pair.

Proof.

Suppose $W = W_e^A = W_i^B$. We need to show that W is c.e. Consider the $\Sigma_2^0(B)$ set

$$X = \{\sigma : \exists x, s \forall t \; x \in W_{e,s}^{\sigma} \text{ and } x \notin W_{i,t}^{B}\}.$$

By hypothesis, A must avoid this set. Let $\sigma \prec A$ be such that $\forall \tau \succ \sigma \; \tau \notin X$. Then W is c.e. since $W = \{x : \exists \tau \succ \sigma \; x \in W_e^{\tau}\}$. □

Answering the original question

Corollary

If X is weak 3-generic, then it is quasiminimal and the total degrees above it do not contain an initial segment of the Turing degrees.

Proof.

Let $Y \geq_e X$ be total. Then Y computes a Δ^0_3 escaping function and hence a 2-generic, the halves of which are mutually 2-generic relative to each other. Then the halves form a minimal pair in the e-degrees and so cannot both lie in the cone above X. $\square$

Questions for Randoms

The analogous question for measure 1 classes would be interesting, as would a table relating levels of randomness with notions of computational strength (escaping would likely be replaced by DNC power).

Question

Is there a quasiminimal such that all total degrees above it compute a 1-random?

There are cones, with total degrees computing effective dimension 1 reals. Negative results would also be of interest. I suspect the following is false.

Question

Is there a quasiminimal such that all total degrees above it compute a PA degree?

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