Separating the halves of an Ahmad pair

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The Enumeration Degrees

We say $X \leq_e Y$ if there is a c.e. operator Γ such that $x \in X \iff \exists \langle x, D \rangle \in \Gamma$ with $D \subset Y$.

1. This is a positive analog of Turing reducibility. One can think of $X \leq_T Y$ as X is $\Delta^0_1(Y)$. Here we can think of $X \leq_e Y$ as X is $\Sigma^0_1(Y^+)$, so that we can only ask positive questions about Y.
2. $\leq_e$ is a preorder on $\mathcal{P}(\omega)$ and the corresponding degree structure induced on $\mathcal{P}(\omega)$ is called the Enumeration degrees denoted by $(\mathcal{D}_e, \leq_e)$.
3. The enumeration jump of a set X is defined by $X' = K_X \oplus \overline{K}_X$ where $K_X := \oplus_{n \in \omega} \Gamma_n(X)$
4. The Turing degrees $(\mathcal{D}_T, \leq_T)$ embed into the enumeration degrees as a partial order under the map $i : A \to A \oplus \overline{A}$ and this embedding respects join and jump. The image of this mapping are the total sets: Sets X for which $\overline{X} \leq_e X$.
5. $\mathcal{D}_T$ and $\mathcal{D}_e$ are both upper semilattices with a least element. The least element in D_T are the computable sets while in $\mathcal{D}_e$ the least element 0_e are the c.e. sets.

Local Structure

Let $\mathcal{D}_T^{\leq 0'}$ be the Turing degrees below $0'_T$ while $\mathcal{D}_e^{\leq 0'}$ denote the enumeration degrees below $0'_e$. Both of these sets form a countable ideal in their respective degree structures.

1. $\mathcal{D}_T^{\leq 0'}$ are precisely the degrees made up of Δ_2^0 sets while $\mathcal{D}_e^{\leq 0'}$ are degrees made up of Σ_2^0 sets.
2. The image of $\mathcal{D}_T^{\leq 0'}$ under i forms a subset of $\mathcal{D}_e^{\leq 0'}$.
3. These two structures are not elementarily equivalent as partial orders: For example Sacks showed that there are minimal Turing degrees below $0'_T$ while Guttridge showed that the enumeration degrees are downward dense.
4. In fact given $X <_e Y \leq 0'_e$ there is a $Z \in (X, Y)$ so that $\mathcal{D}_e^{\leq 0'}$ is actually dense.

Ahmad pairs

In her thesis, Ahmad constructs a pair (A, B) of Δ^0_2 sets such that $A \not\leq_e B$ but $\forall Z <_e A (Z <_e B)$. Such pairs were later on named Ahmad pairs by the community. Ahmad in Ahmad and Lachlan [1998] also shows that if (A, B) form an Ahmad pair, then (B, A) cannot be an Ahmad pair.

1. This phenomenon is unique to the Σ^0_2 enumeration degrees and cannot occur in the c.e. or the Δ^0_2 Turing degrees.
2. Slaman and Soare solved the extension of embeddings problem in the c.e. Turing degrees where given finite partial orders $\mathcal{P} \subset \mathcal{Q}$, when every embedding of $\mathcal{P}$ can be extending to an embedding of $\mathcal{Q}$. They provide 2 obstructions barring which an extension is always possible.
3. In Lempp et al. [2005] solve the extension of embeddings problem for the Σ^0_2 enumeration degrees, where the only added obstruction to extension is the phenomenon of Ahmad pairs.

Recently several researchers have been focusing on solving the $\forall\exists$ theory of the Σ^0_2 enumeration degrees. This has resulted in renewed interest in Ahmad pairs with the following two recent results:

1. In Goh et al. [2022] the authors extend Ahmad's result to show that if (A, B) is an Ahmad pair, then (B, C) cannot be an Ahmad pair for any C. This shows that the right and left halves are disjoint.
2. They also construct a so called weak Ahmad triple, sets (A, B_0, B_1) such that A is not the left half of an Ahmad pair, $A|_e B_0, B_1$ and $\forall Z <_e A$ we have $Z \leq_e B_0$ or $Z \leq_e B_1$.
3. In Kalimullin et al. [2024] the authors show that if (A, B) form an Ahmad pair, then $A \oplus B <_e 0'$.

In this talk I will present recent results extending and generalizing some of the work above.

The main result we have is the following:

Definition

The tuple $(A, B_0, ..., B_{n-1})$ forms an Ahmad n pair if $A|_e B_i \forall i < n$ and for any $Z <_e A$ there is an $i < n$ such that $Z \leq_e B_i$.

Theorem

If $(A, B_0, B_1, ..., B_{n-1})$ form an Ahmad n pair, then $A \in \mathrm{low}_3$ and $\oplus_{i<n} B_i \notin \mathrm{low}_3$.

Corollary

If (A, B) form an Ahmad pair, then $A \in \mathrm{low}_3$ and $B \notin \mathrm{low}_3$. Therefore the left and right halves are disjoint.

The left half

Note that if (A, B) form an Ahmad pair, then the set $\{Z : Z <_e A\}$ is an ideal. In particular A is join irreducible.

Theorem

Let f, g be computable. There is a computable function h such that $\{\Gamma_{h(n)}(0'_e)\}_n = \{\Gamma_{f(n)}(0'_e)\}_n \cap \{\Gamma_{g(n)}(0'_e)\}_n$

Proof.

Recall that the Σ^0_2 sets are precisely those which are $c.e.$ relative to $0'$. So let $\{W^{0'}_{f(n)}\}_n$ and $\{W^{0'}_{g(n)}\}$ be the uniform families of Σ^0_2 sets. Then we define $h(n)$ as follows:

$$x \in W^{0'}_{h(\langle e,i \rangle),s} \iff x \in W^{0'}_{f(e),s} \cap W^{0'}_{g(i),s} \text{ and } W^{0'}_{f(e),s} \restriction_x = W^{0'}_{g(i),s} \restriction_x .$$

□

The left half

Theorem

If $(A, B_0, ..., B_{n-1})$ form an Ahmad n pair, the ideal $\{Z : Z <_e A\}$ has a uniform enumeration.

Proof.

By the parameter theorem the ideals $\{Z : Z \leq_e A\}$ and $\{Z : Z \leq_e B_i\}$ have uniform enumerations. Therefore so does their intersection $\mathcal{F}_i := \{Z : Z \leq_e A, B_i\}$. Then $\mathcal{F} = \cup_{i<n}\mathcal{F}_i$ has a uniform enumeration as well and $\mathcal{F} = \{Z : Z <_e A\}$. □

We call a uniform enumeration of $\{Z : Z <_e A\}$ an Ahmad sequence for A.

Theorem

The following are equivalent:

1. *A has an Ahmad sequence $\{\Gamma_{f(n)}(0'_e)\}$.*
2. *$X = \{n : \Gamma_n(A) <_e A\}$ is Δ^0_4.*
3. *A is low_3.*

The left half

Proof.

$(1 \implies 2)$ $X \leq_e A^{\langle 3 \rangle}$ and so is always Π^0_4. Using the Ahmad sequence it also has a Σ^0_4 definition.
$(2 \implies 3)$ We will show that $A^{\langle 3 \rangle} \leq_e X$ below. Then $A^{\langle 3 \rangle} \in \Sigma^0_4$ so $A^{\langle 3 \rangle} \leq_e 0^{\langle 3 \rangle}$.
$(3 \implies 1)$ Note that $X \leq_e A^{\langle 3 \rangle}$ and so X has a Σ^0_4. Let

$$e \in X \iff \exists n \forall m \exists i \forall j R(e, n, m, i, j)$$

Then using this we can define an Ahmad sequence $\{A_{e,n}\}$ for a such that if $n \in X$ then $\exists e(A_{e,n} = \Gamma_n(X))$ while if $n \notin X$ then $A_{e,n}$ is finite for every e. We define $A_{e,n}$ as a Σ^0_2 approximation which agrees with $\Gamma_n(X)$ on even stages and at odd stages we put in $0/1$ according to the Σ_4 definition, details on board. □

The left half

Lemma

There is a computable function g *such that* $\Gamma^{[i]}_{g(e)}$ *is either* ω *or a finite initial segment of* ω *for every* $e, i \in \omega$ *and:*

1. $e \in A^{\langle 3 \rangle} \iff \Gamma^{[i]}_{g(e)}(A)$ *is a finite initial segment of* ω *for every* i
2. $e \notin A^{\langle 3 \rangle} \iff \exists i(\Gamma^{[i]}_{g(e)}(A) = \omega)$

Proof.

Let $e \in A \iff \forall i \exists j \forall k R(e, i, j, k)$ where $\neg R(e, i, j, k) \leq_e A$. Then define g as follows:

Given an e, let X_e be the set where we enumerate j into $X_e^{[i]}$ if $\forall j' < j$ we find a k with $\neg R(e, i, j', k)$. Then $X_e \leq_e A$ and by construction if $j_0 < j_1$ and $j_1 \in X_e^{[i]}$ then $j_0 \in X_e^{[i]}$. Since we can go from $e \to X_e$ uniformly, there is a computable g such that $X_e = \Gamma_{g(e)}(A)$.

□

The left half

Definition

Let X, Y be Σ_2 sets.

1. A good approximation to X is a computable sequence $\{X_s\}_s$ of finite sets with infinitely many good stages $G_X := \{s : X_s \subset X\}$ such that $\lim_{s \in G_X} X_s(n) = X(n)$ for every n.
2. A correct approximation $\{Y_s\}$ to Y with respect to a good approximation $\{X_s\}$ to X is an approximation where $G_X \subset G_Y$ and $\lim_{s \in G_X} Y_s(n) = Y(n)$.

Lemma

For any set $X \leq_e A$ with A non c.e. such that $\forall i \exists j \leq \omega (X^{[i]} = \omega \restriction_j)$ we can uniformly build an enumeration operator Θ such that $\Theta(A) <_e A \iff X^{[i]}$ is finite for every i and $\Theta(A) \geq_e A \iff \exists i (X^{[i]} = \omega)$.

Corollary

$A^{\langle 3 \rangle} \leq_e \{e : \Gamma_e(A) <_e A\}$.

The left half

Proof.

Let $X = \Delta(A)$ and let $\{A_s\}_s$ be a good approximation to A. We shall build the enumeration operator Θ to meet the requirements:

$$\mathcal{R}_e : \Gamma_e(\Theta(A)) \neq A \iff X^{[\leq e]} \text{is finite}$$

At stage $s = 0$ let $\Theta = \emptyset$.
At stage $s + 1$, we have substages $t \leq s$:
At substage t we do the following:

1. Let $l_{t,s} = l(\Gamma_{t,s}(\Theta(A_s)), A_s)$. Then $\forall x \leq l_{t,s}$ with $x \in A_s$, let $\langle x, D\rangle \in \Gamma_{t,s}$ be the least axiom witnessing this. Then for every $y \in D^{[\geq t]}$, add the axiom $\langle y, A_s\rangle$ into Θ.
2. Copy $A \restriction_n$ where $n = |\Delta_s^{[t]}(A_s)|$ into the t^{th} column by enumerating axioms $\langle\langle t, x\rangle, \{x\} \cup A_s\rangle$ into Θ for every $x \leq n$.

This ends the construction. □

Characterizing the left half

Definition

A is join n irreducible if for every $A_0, .., A_n <_e A$ there is an $i, j \leq n$ with $A_i \oplus A_j <_e A$.

Theorem

A is the left half of an Ahmad n pair $\iff$ A is low_3 and join n irreducible.

Proof.

If A is the left half of an Ahmad n pair it has an Ahmad sequence and is therefore low_3. It is easy to see that it must be join n irreducible.
For the converse we need the following lemma along with induction on n (details on board). □

Lemma

Let f be computable and $\mathcal{F} = \{\Gamma_{f(n)}(A)\}_n$ be an ideal such that $\forall n(A \not\leq_e \Gamma_{f(n)}(A))$. Then there is a Σ^0_2 set B with $A \not\leq_e B$ and $\forall X \in \mathcal{F}(X \leq_e B)$.

Proof.

We will build a B by coding $\Gamma_{f(n)}(A)$ into the n^{th} column of B while ensuring that $A \not\leq_e B$. Let $\{A_s\}_s, \{B_{n,s}\}_s$ be correct approximations to $A, \Gamma_{f(n)}(A)$ respectively with respect to a good approximation K_s to $\overline{K}$. We will build an enumeration operator Θ so that $B = \Theta(\overline{K})$ will meet the requirements:

$$\mathcal{N}_e : A \neq \Gamma_e(B)$$

$$\mathcal{P}_n : \Gamma_{f(n)}(A) \leq_e B^{[n]}$$

At stage $s = 0$, let $\Theta = \emptyset$.
At stage $s + 1$:

1. For $e \leq s$ let $l_{e,s} = l(A_s, \Gamma_{e,s}(B_s))$. Then $\forall x < l_{e,s}$ with $x \in \Gamma_{e,s}(B_s)$, pick the least axiom $\langle x, D\rangle \in \Gamma_e$ which witnesses this. Now for all $y \in D^{[>e]}$ enumerate the axioms $\langle y, K_s\rangle$ into Θ.
2. For $n \leq s$ if $x \in B_{n,s+1} = \Gamma_{f(n),s+1}(0'_e)$ then for every new axiom $\langle x, D\rangle \in \Gamma_{f(n),s+1} - \Gamma_{f(n),s}$ enumerate the axiom $\langle\langle n, x\rangle, D\rangle$ into Θ.

□

On the right half

Definition

A set G is $A-$ Guttridge if there is a computable function f such that $f(x,.)$ is increasing, $\lim_s f(x,s)$ exists for every x and $\langle x,y\rangle \in G \iff \exists s(f(x,s) > y$ or $f(x,s) = y$ and $x \in A)$.

Lemma

Suppose (A,B) form an Ahmad pair. Then $A^\diamond \leq_e B^\diamond$

Proof.

This is implict in Ahmad's no symmetric Ahmad pair argument. Let $G <_e A$ be a K_A Guttridge set with f being the witnessing computable function. Then $G \leq_e B$ and so $K_A \leq_e B \oplus 0'_e$ and $\overline{K}_A \leq \overline{B} \oplus 0'_e$ and hence $A^\diamond \leq_e B^\diamond$. □

Theorem

Suppose $B \in \mathrm{low}_3$. Then $\forall A$ such that $A \not\leq_e B$ we can build an enumeration operator Θ such that $\Theta(A) <_e A$ and $\Theta(A)|_e B$.

On the right half

Proof.

Suppose $\{\Theta_n\}_{n\in\omega}$ are a family of enumeration operators. Consider the statement $\Theta_n(A) \not\leq_e B$ for any n:

$$\forall n, m \exists x (x \in \Theta_n(A) \wedge x \notin \Gamma_m(B)) \text{or} (x \in \Gamma_m(B) \wedge x \notin \Theta_n(A))$$

This statement is $\leq_e B^{\langle 3 \rangle}$ and is Σ_4 if B is low_3. Let $\exists n S_n$ where S_n is Π^0_3 be a Σ_4 definition of the statement above.
We construct a Θ such that its columns $\Theta^{[n]}$ correspond to Θ_n above. By the recursion theorem, we may assume we know an index for Θ and so while constructing Θ we can reason about the statement $\exists n S_n$. We will ensure that the following holds:

1. $S_n \implies \Theta_n(A) <_e A$
2. $\neg S_n \implies \Theta_n(A) \equiv_e A$

□

Proof.

Consider the following cases:

1. The statement is false: $\exists n(\Theta_n(A) \leq_e B)$. Then $\forall n \neg S_n$ and so by construction $\Theta_n(A) \equiv_e A$ for every n, a contradiction to $A \not\leq_e B$.
2. The statement is true: $\forall n(\Theta_n(A) \not\leq_e B)$. The we also have $\exists n S_n$ and so $\Theta_n(A) <_e A$ for some n and we are done!

To construct the Θ_n's the the property that $S_n \iff \Theta_n(A) <_e A$ we just need to use the lemma above! □

We end with some questions:

1. Does the right half have to be high? (It is known that not all high bounds a right half)
2. Is there a simpler proof of the fact that Ahmad pairs do not cup to $0'_e$?
3. Do each of the right halves of Ahmad n pairs have to be non low_3?

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